Quarterly Journal of
Optoelectronic
Year 2, No. 3 (Serial Number 7) Winter 2018 (P 53-60)

فصلنامه
اپتوالکترونیک
سال دوم، شماره سوم (پیاپی 7)، زمستان 1396 (ص 53 - 60)

رفتار فلاکسون‌های اتصالات جوزفسون در یک محیط ناهمگن

سارا خسروانی *
کارشناسی ارشد، فیزیک، دانشگاه پیام نور

تاریخ دریافت: 1397/08/16 **تاریخ پذیرش**: 1398/07/02

Fluxon Behavior of Josephson Junction in Inhomogeneous Media

S. Khosravani*
Ms.C., Physics, Payame Noor University

Received: 2018/11/07 **Accepted:** 2019/08/24

Abstract
If the dielectric between two superconductors of the Josephson junction is homogeneous the governing equation for the phase of the system can be described by the sine-Gordon equation. In this paper, we consider inhomogeneous dielectric medium, such that the dielectric constant is position dependent. Then, we derive the sine-Gordon equation correspondence of this system and find its analytical and numerical soliton solutions. We simulate the particle behavior of the soliton solutions, and finally, we show that they obey the Newtonian's second law of motion.

Keywords
Soliton, Sine-Gordon Equation, Josephson Junction.

چکیده
اگر دی‌الکتریک بین دو ابررسانای اتصال جوزفسون همگن باشد، معادله حاکم بر این سیستم، معادله سینوسی گوردون است. در این مقاله حالتی که در آن دی‌الکتریک بین دو ابر رسانا ناهمگن و تابع مکان باشد را مورد بررسی قرار می‌دهیم. سپس پاسخ تحلیلی و عددی آن را به دست می‌آوریم و آن را تجزیه و تحلیل کرده و رفتار ذره را به کمک پاسخ سالیتونی شبیه‌سازی می‌کنیم و سرانجام نشان می‌دهیم که این پاسخ جایگزیده از قانون دوم نیوتن تبعیت می‌کند.

واژگان کلیدی
سالیتون، معادله سینوسی گوردون، اتصال جوزفسون.

*Corresponding Author: damoon.b@gmail.com

*** نویسندهٔ مسئول**: سارا خسروانی

Introduction

One of the most successful testing grounds for nonlinear wave theory is the Josephson transmission line. Such transmission lines are used for information processing and storage. In a long Josephson junction or transmission line, the physical quantity of interest is a quantum of magnetic flux, or a fluxon, which has a soliton behavior. Consequently, it can be used as a basic bit in information processing systems. It can be shown that the governing equation for this phenomenon is the sine-Gordon equation [1, 2]. The tunneling effect of Cooper pairs across a thin insulator between two superconductors was predicted by B. D. Josephson [3-6]. If two superconductors are separated by a thin insulator or non-superconductor, super current can flow from one superconductor to another; this is known as Josephson effect [7].

When two superconductors are separated by a distance d , if d is large, the wave functions and phases of two superconductors are independent. If d is small, single electrons can flow from one superconductor to another with tunneling phenomenon. If d is very small about 30 A^o, cooper pair can flow from one superconductor to another. Josephson discovered if two superconductors are separated by very small layer insulator, cooper pair can be tunneling from one superconductor to another and this junction is called Josephson junction. In Josephson junction some special and interesting phenomena can occur. The most important one is to appear fluxons.

Fluxon is a circulating super current across the insulator layer Josephson junction.

If a bias current applied to a fluxon, it can be forced to move along the junction, when circulating current induces a magnetic filed, the fluxon can endure a Lorenz force, thus the fluxon starts to move along junction with increasing speed until it reaches the maximum speed that is balanced between the dissipative effects and Lorenz force fluxon in Josephson junction.

In sec. 2 we will derive sine-Gordon equation that governs Josephson junction. Any spatial variation in the dielectric permeability e or variation in the separation between two superconductors, results in a position-dependent of capacitance and inductance. This -in turn- affects the propagation of kinks in the junction. The situation can be approximated by the classical analog of a point particle moving in a velocity dependent external potential. We will discuss this further in sec 3. In sec 4 we will discuss numerical calculation and give some example. Finally, we conclude our discussion with conclusion and results.

Derivation of Sine-Gordon Equation in Josephson Junction

Electrons can move as Cooper pairs in each superconductor. If the common macroscopic wave function of all the electron pairs is written as,

$$y = \sqrt{R} e^{if} , \qquad [1]$$

where R is the Cooper pair density and f is the quantum phase common to all the pairs, in this case, the two superconductors will naturally have independent wave functions y_1 and y_2 with uncorrelated phases f_1 and f_2, unless the two superconductors are set near enough to each other, say less than about 30 A^o. See Figure 1. The phases then become correlated because of Cooper pair penetration through the insulator barrier.

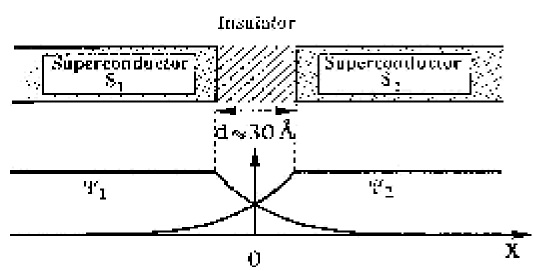

Fig 1. Two coupled superconductor with weak link and amplitude of macroscopic wave function of two superconductors.

The wave functions y_1 and y_2 satisfy two coupled linear Schrodinger equations [8]

$$ih\frac{¶y_1}{¶t} = E_1 y_1 + k y_2 , \qquad (2)$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = E_2\psi_2 + k\psi_1 \qquad (3)$$

where E_1 and E_2 are the ground state energies of electrons in the two superconductors. Here, we have assumed that the two superconductors are similar. k is a real coupling constant which depends on the characteristics of the junction. Obviously, if $d \to \infty$ then $k \to 0$, where d is the barrier thickness. When a static potential difference V is maintained between the two superconductors, an energy shift $E_1 - E_2 = 2eV$ is developed. We can arbitrarily choose the reference energy at $E = (E_1 + E_2)/2 = 0$, and therefore $E_1 = eV$ and $E_2 = -eV$. Equations (2) and (3) then respectively become

$$i\hbar\frac{\partial\psi_1}{\partial t} = eV\psi_1 + k\psi_2, \qquad (4)$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = -eV\psi_2 + k\psi_1. \qquad (5)$$

Using the expressions $\psi_1 = \sqrt{R_1}e^{i\phi_1}$ and $\psi_2 = \sqrt{R_2}e^{i\phi_2}$ in these equations and separating the real and imaginary parts, we obtain

$$\hbar\frac{\partial R_1}{\partial t} = -2k\sqrt{R_1R_2}\sin\phi, \qquad (6)$$

$$\hbar\frac{\partial R_2}{\partial t} = +2k\sqrt{R_1R_2}\sin\phi, \qquad (7)$$

$$\hbar\frac{\partial\phi_1}{\partial t} = k\sqrt{R_2/R_1}\cos\phi - eV \qquad (8)$$

$$\hbar\frac{\partial\phi_2}{\partial t} = k\sqrt{R_1/R_2}\cos\phi + eV, \qquad (9)$$

in which $\phi = \phi_2 - \phi_1$ is the phase difference between the two wave functions.

Let us define the quantities $J_1 = \partial R_1/\partial t$ and $J_2 = \partial R_2/\partial t$. R_1 and R_2 represent electron pair densities which deviate only slightly from their equilibrium values R_0. We therefore have $R_1 \approx R_2 \approx R_0$, and $(2k/\hbar)\sqrt{R_1R_2} \approx 2kR_0/\hbar = J_0$, and therefore

$$J \approx J_0\sin\phi, \qquad (10)$$

according to (6) or (7). By subtracting Equation (8) from (9) yield

$$\hbar\frac{d\phi}{dt} = 2eV. \qquad (11)$$

We can write equation (11) in the form

$$\frac{d\Phi}{dt} = V, \qquad (12)$$

where Φ has the dimensions of magnetic flux, and is defined according to

$$\phi = 2\pi\frac{\Phi}{\Phi_0}, \qquad (13)$$

in which $\Phi_0 = h/2e = 2.064 \times 10^{-15}$Wb is the quantum of magnetic flux. From (10) and (13) we have

$$\Phi = \frac{\Phi_0}{2\pi}\sin^{-1}\frac{J}{J_0}. \qquad (14)$$

In practice, this nonlinear flux-current

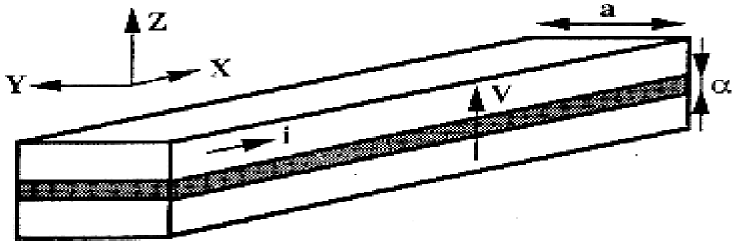

Fig. 2. Long Josephson junction

relation can be thought of as representing a nonlinear inductance.

If $V = 0$, then (11) implies $\phi = \text{const}$. which is in general non-vanishing. Then from (10) this leads to a finite current density J even in the absence of an applied voltage. This effect is known as DC Josephson effect. If $V = V_0 = \text{const.}$, $\Phi = V_0 t + \Phi_1$ where Φ_1 is a constant of integration, and (14) yields an alternating current density which is known as AC Josephson effect.

$$J = J_0 \sin \frac{2\pi}{\Phi_0}(V_0 t + \Phi_1). \tag{15}$$

Therefore, an alternating current density develops with an angular frequency

$$\omega_J = \frac{2\pi V_0}{\Phi_0} = \frac{2eV_0}{\hbar}. \tag{16}$$

This frequency is of the order of a few hundred MHz per μV voltage difference [9-11].

We now turn to a long Josephson junction, which consists of two relatively long strips of superconducting materials, separated by a very thin dielectric of thickness d.

According to Figures (2) a length element dx of this device, is electrically equivalent to the circuit with capacitance per unit length

$$C = \frac{K\varepsilon_0 a}{d}. \tag{17}$$

in which K is the dielectric constant of the dielectric, $\varepsilon_0 = 8.854 \times 10^{-12} C^2/Nm^2$ and a is the width of the superconducting strip. Inductance per unit length is

$$L = \mu_0 \frac{2\lambda_L + d}{a}, \tag{18}$$

where $\mu_0 = 4\pi \times 10^{-7}$, and λ_L is the penetration depth of the superconductors[9].

From basic circuit theory, the following equations result

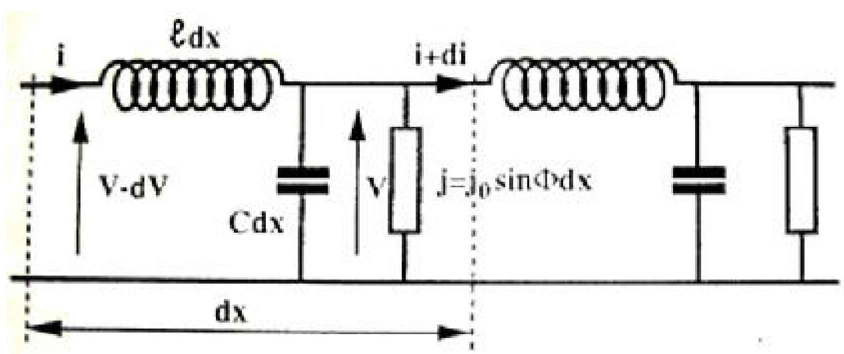

Fig. 3. equivalent circuit of a Josephson junction

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \tag{19}$$

and

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - J_0 \sin 2\pi \frac{\Phi}{\Phi_0}, \tag{20}$$

$$\frac{\partial \Phi}{\partial t} = V. \tag{21}$$

Equation (19) can be derived by using the Kirshohf's voltage law and equation (20) by Kirshohf's current law [Grant and Philips, 1975]. It should be noted that the source of nonlinear sine term in equation (20) is from superconductivity of the strip. These equations can be easily combined to yield the following sine-Gordon equation for the phase difference [12, 13]

$$\frac{\partial^2\phi}{\partial t^2} - c_J^2\frac{\partial^2\phi}{\partial x^2} + \omega_p^2\sin\phi = 0, \qquad (22)$$

in which

$$c_J = \frac{1}{\sqrt{LC}}, \text{ and } \omega_p = \sqrt{\frac{2\pi J_0}{\Phi_0 C}}. \qquad (23)$$

And (14) has been used. Note that c_J/ω_p has dimensions of length. It describes a length scale, called Josephson penetration length. This determines whether a Josephson junction is 'long' or not [14].

Equation (22) can obviously have the kink solution, in which $\beta = 1$, $a = \omega_p^2/c_J^2$ and $c \to c_J$. The corresponding current I, and voltage V, can then be easily calculated using equations (19), (20) and (21). The kink (anti-kink) describes a pulse of 2π (-2π) phase difference, corresponding to a quantum of magnetic flux accompanied by a voltage and current pulse. The kink (anti-kink) is thus called a fluxon (anti-fluxon) in this case [15].

Sine-Gordon Equation of Josephson Junction in Inhomogeneous Media

If we assume permeability (ε) to be position dependent ($\varepsilon = \varepsilon(x)$), then the sine–Gordon equation in inhomogeneous media is:

$$\frac{\partial^2\varphi}{\partial x\partial t} = -l(x)\frac{\partial i}{\partial t}, \qquad (24)$$

$$\frac{\partial i}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{1}{l(x)}\frac{\partial\varphi}{\partial x}\right), \qquad (25)$$

$$\frac{\partial i}{\partial x} = -C(x)\frac{\partial^2\varphi}{\partial t^2} - J_0\sin\varphi. \qquad (26)$$

If we insert Eq. (25) into Eq. (26), we obtain:

$$\frac{\partial l^{-1}(x)}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{1}{l(x)}\frac{\partial^2\varphi}{\partial x^2} - C(x)\frac{\partial^2\varphi}{\partial t^2} = J_0\sin\varphi. \qquad (27)$$

Using a little algebra, we can find:

$$\varphi_{xx} - \frac{\varepsilon(x)}{C_0^2}\varphi_{tt} = \frac{1}{\lambda_J^2}\sin\varphi, \qquad (28)$$

where $\lambda_J = \sqrt{\frac{\Phi_0}{2\pi\mu_0 J_0 d}}$ and $C_0 = \frac{k\varepsilon_0 a}{d}$.

In order to solve this Eq., we assume:

$$\varphi(x) = 4\tan^{-1}[\exp(\alpha(x))(x - x_k(t) - x_0)], \qquad (29)$$

Where $\alpha(x)$ and $x_k(t)$ are unknown functions. For calculating the differential Eq. governed on $\alpha(x)$ and $x_k(t)$ we must differentiate the Eq. (29) with respect to x and t twice, we have

$$\frac{\partial^2\varphi}{\partial x^2} = 2f''\operatorname{sech} f - 2f'^2\operatorname{sech} f\tanh f = 2f''\sin\frac{\varphi}{2} + f'^2\sin\varphi, \qquad (30)$$

$$\frac{\partial^2\varphi}{\partial t^2} = 2\ddot{f}\operatorname{sech} f - 2\dot{f}^2\operatorname{sech} f\tanh f = 2\ddot{f}\sin\frac{\varphi}{2} + \dot{f}^2\sin\varphi, \qquad (31)$$

where

$$f(x,t) = \alpha(x)(x - x_k(t) - x_0). \qquad (32)$$

Substituting these relations in Eq. (28), we obtain

$$2f''\sin\frac{\varphi}{2} + f'^2\sin\varphi - \frac{\varepsilon(x)}{C_0^2}\left(2\ddot{f}\sin\frac{\varphi}{2} + \dot{f}^2\sin\varphi\right) - \frac{1}{\lambda_J^2}\sin\varphi = 0. \qquad (33)$$

Equating the coefficients $\sin\varphi$ and $\sin\frac{\varphi}{2}$ then,

$$f'^2 - \frac{\varepsilon(x)}{C_0^2}\dot{f}^2 - \frac{1}{\lambda_J^2} = 0, \qquad (34)$$

$$2f'' - \frac{2\varepsilon(x)}{C_0^2}\ddot{f} = 0. \qquad (35)$$

By substituting f', f'', $\dot{f}$ and $\ddot{f}$ in Eq.s (30) and (31) respectively, thus we obtain

$$2\alpha''(x)(x - x_k(t)) + 4\alpha'(x) + \frac{2\varepsilon(x)}{C_0^2}(\alpha(x)\ddot{x}_k(t)) = 0, \qquad (36)$$

$$(\alpha'(x)(x - x_k(t)) + \alpha(x))^2 - \frac{\varepsilon(x)}{C_0^2}(\alpha(x)\dot{x}_k(t))^2 - \frac{1}{\lambda_J^2} = 0. \qquad (37)$$

In order to determine the dynamics of the center of mass of the soliton we put $x = x_k(t)$ in Eq. (36) and (37), which leads to,

$$\alpha(x_k(t))\ddot{x}_k(t) = -\frac{2C_0^2}{\varepsilon(x_k(t))}\alpha'(x_k(t)), \qquad (38)$$

$$\alpha^2(x_k(t))\dot{x}_k^2(t) = \frac{C_0^2\alpha^2}{\varepsilon(x_k(t))} - \frac{C_0^2}{\varepsilon(x_k(t))\lambda_J^2}. \qquad (39)$$

Multiplying Eq. (38) by $\dot{x}_k(t)$ and integrating it with respect to t, we obtain

$$\alpha(x_k(t))\frac{\dot{x}_k(t)}{2} = -\frac{2C_0^2}{\varepsilon(x_k(t))}\alpha(x_k(t)) + C \qquad (40)$$

Eq. (39) is divided into 2α then:

$$\alpha(x_k(t))\frac{\dot{x}_k^2(t)}{2} = \frac{C_0^2\alpha(x_k(t))}{2\varepsilon(x_k(t))} - \frac{C_0^2}{2\varepsilon(x_k(t))\lambda_J^2\alpha(x_k(t))}, \qquad (41)$$

Left side of the Eq. (40) is equal to Eq. (41), then we can obtain integration constant, C,

$$C = \frac{2\alpha(x_k(t))C_0^2}{\varepsilon(x_k(t))} + \frac{C_0^2\alpha(x_k(t))}{2\varepsilon(x_k(t))} - \frac{C_0^2}{2\varepsilon(x_k(t))\lambda_J^2\alpha(x_k(t))}. \qquad (42)$$

As C is constant, we derive it for special case $\alpha = \alpha_0$ and $\varepsilon(x) = \varepsilon_0$ then,

$$C = \frac{5\alpha_0 C_0^2}{2\varepsilon_0} - \frac{C_0^2}{2\lambda_J^2\varepsilon_0\alpha_0}. \qquad (43)$$

Substituting C into Eq. (42) we can obtain:

$$\alpha(x) = \frac{\alpha_0\varepsilon(x)}{2\varepsilon_0} - \frac{\varepsilon(x)}{10\lambda_J^2\varepsilon_0\alpha_0} + \sqrt{A}, \qquad (44)$$

Where

$$A = \left(\frac{\alpha_0\varepsilon(x)}{2\varepsilon_0} - \frac{\varepsilon(x)}{10\lambda_J^2\varepsilon_0\alpha_0}\right)^2 + \frac{1}{5\lambda_J^2}.$$

In order to confirm our calculation, If $\lambda_J = 1$ and $\varepsilon(x) = \varepsilon_0 = 1$ then $\alpha = \alpha_0$. Rescaling our equation as $\alpha_0 = 1$, $\lambda_J = 1$ and $\varepsilon_0 = 1$ then

$$\alpha(x) = \frac{4}{10}\varepsilon(x) + \sqrt{\frac{16}{100}\varepsilon^2(x) + \frac{1}{5}}.$$

If $\varepsilon(x) = 1 + V(x)$ and assume $V(x)$ is very small then it yields $\alpha \approx 1 + \frac{2}{3}V(x)$

From Eq. (38), we can obtain the dynamics of the soliton's center of mass as:

$$\frac{\varepsilon(x_k(t))}{C_0^2}\frac{d^2 x_k(t)}{dt^2} = -\frac{2\alpha'(x_k(t))}{\alpha(x_k(t))} = -\frac{d}{dx}(2\ln\alpha(x_k(t)). \qquad (45)$$

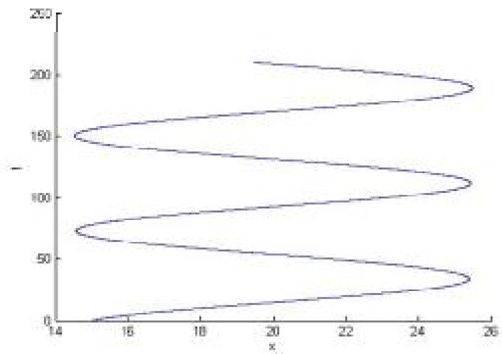

Fig. 4b. Time evolution of the center of mass of the soliton in harmonic potential.

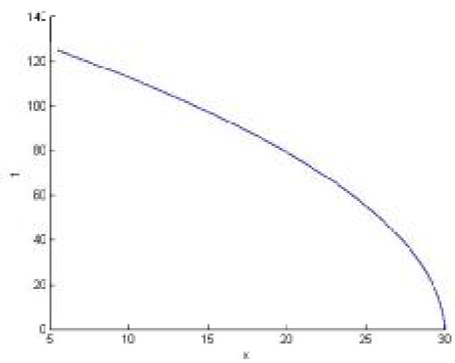

Fig. 5b. Time evolution of the center of mass of the soliton in the potential of inclined

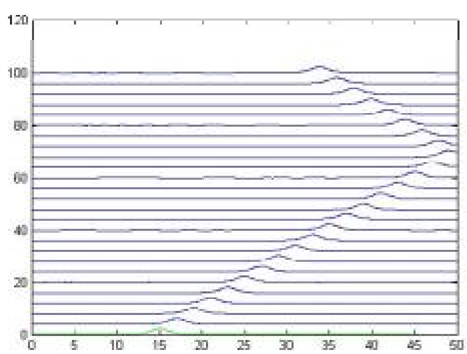

Fig. 6b. Time evolution of a soliton against a potential barrier. The soliton can go through the barrier.

Comparing this equation with Newton's

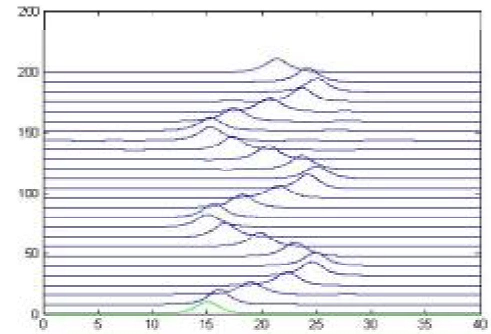

Fig. 4a. Time evolution of a soliton against in harmonic potential.

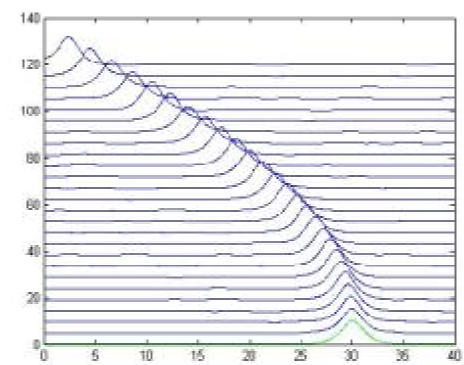

Fig. 5a. Time evolution of a soliton against the potential of inclined plane.

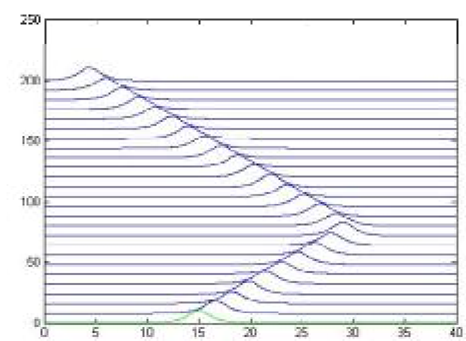

Fig. 6a. Time evolution of a soliton against a potential barrier. It is reflected from barrier.

Second law

$$m\frac{d^2x}{dt^2} = -\frac{dU}{dx}, \qquad (46)$$

Thus

$$m = \frac{\varepsilon(x)}{C_0^2}\ ,\ U = 2\ln\alpha\ . \qquad (47)$$

In this equation, m is the center of mass soliton and $U = 2\ln(1+\frac{2}{3}V(x))$ is potential of soliton.

If variable of the dielectric constant is small $V(x) \ll 1$ then $U \approx \frac{4}{3}V(x)$ and $m = \varepsilon(x) = \varepsilon_0(1+V(x)) \approx 1$.

Then

$$\&\&\& + \frac{4}{3}\frac{dV}{dx} = 0\ . \qquad (48)$$

Numerical Results

We solve Eq. (28) by method of finite difference for $\varepsilon(x) = \varepsilon(1+V(x))$ and Eq. (48), and campare them. In all the figures (4a), (5a) and (6a), we have considered Eq. (28), and draw the time evolution of the derivative of the soliton solution. In figures (4b), (5b) and (6b), we simultaneously solve Eq. (48) numerically with the same initial conditions. We can draw time evolution of the center of mass of the soliton. By comparing these two figures, we observe that they are completely identical.

Conclusions

In this paper, we have changed Josephson junctions and taken the coefficient of dielectric as a function of position, and then we have solved sine-Gordon eq. in this media. At last, we have numerically solved the difference potential and observed that behavior of solitons conformed to the Newton's second law. Thus, we can control departure of fluxon and make new electromagnetic keys.

References

[1]. W. Perold, “Superconducting Quantum Interference Device (SQUID) Mangnetometers: Principles, Fabrication and Applications”, Stellenbosch University, (2010).
[2]. R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on physics, Addison_Wesley, Reading, Massachusetts, Vol. III, (1966).
[3]. R. Rajaraman, “Solitons and Instantons”, Elsevier, North Holland, (1996).
[4]. A. Wallraff aus Bonn, “Fluxon Dynamics in Annular Josephson Junction from Relativistic strings to Quantum particles”, (2000).
[5]. G. Grosso, “Solid State Physics”, University of Pavia, Academic Press, (2000).
[6]. N. Anderson and K. Mortenson, “Superconductivity”, Physics department, Academic Press, (1999).
[7]. H. Kooiker, “Fluxon Dynamics of the State in a Two-Fold Stack of Long Josephson Junctions”, University of Twente, Enschede, (2005).
[8]. T. Visser, “Modelling and Analysis of Long Josephson junction”, PhD thesis, University of Twente, (2002).
[9]. N. Riazi, “Dynamics of Salitons in Inhomogeneous Josephson Junctions”, Int. J. Theo. Phys., Vol. **35**, No. 1, (1996).
[10]. A. Wallraff, “Fluxon Dynamics and Radiation Emission in Two Fold Long Josephson Junction Stacks”, American Institute of Physics, (1996).
[11]. A. Abdumalikov, “Vortex Dynamics in Ultra-Narrow Josephson Junctions”, Lehrstuhl für Mikrocharakterisierung, Friedrich-Alexander-Univ., (2005).
[12]. N. Riazi, “Dynamics of salitons in Inhomogeneous Josephson Junctions”, International Journal of Theoretical Physics, Vol. **35**, No. 1, (1996).
[13]. N. Riazi and A. R. Gharaati, “Dynamics of sine-Gordon Solitons”, Int. J. Theor. Phys., Vol. **37**, No. 1, (1998).
[14]. A. V. Ustinov, “Long Josephson Junctions and Stacks”, Physikalisches Institute III, Universitat Erlangen-Nurnberg D-91054, Erlangen, Germany, (1998).
[15]. M. Remoissent, “Wave called Solitons”, Springer Verlag Berlin, 2nd edition, (1996).